Artículo

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A New Generalized Matrix Inverse Method for Balancing Chemical Equations and their Stability

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The matrix method can be used for any type and complexity of system.
Smith, W. R.; Missen, R. W. *J. Chem. Educ*. **1997**, *74*, 1371.

Abstract. In this work is given a new generalized matrix inverse method for balancing chemical equations. Here offered method is founded by virtue of the solution of a homogeneous matrix equation by using of von Neumann pseudoinverse matrix. The method has been tested on many typical chemical equations and found to be very successful for the all equations in our extensive balancing research. The method works successfully without any limitations. Chemical equations treated here possess atoms with fractional oxidation numbers. Also, in the present work are analyzed some necessary and sufficient criteria for stability of chemical equations over stability of their reaction matrices. By this method is given a formal way for balancing general chemical equation with a matrix analysis.
Key word: Mathematical method, matrices, balancing chemical equations, stability.

Resumen. En este trabajo se presenta un nuevo método generalizado de matriz inversa para el balanceo de ecuaciones químicas. El método se basa en la solución de una matriz homogénea de ecuaciones usando la matriz pseudoinversa de von Neumann. El método se ha probado en muchas ecuaciones químicas típicas y se encontró de gran utilidad para todas las ecuaciones en una investigación extensiva. El método funciona apropiadamente y no tiene limitaciones. Las ecuaciones químicas mostradas aquí poseen números de oxidación fraccionarios. También se analizan algunos criterios suficientes y necesarios para la estabilidad de las ecuaciones químicas sobre la estabilidad de sus matrices de reacción. Por este método se da una manera formal de balanceo de ecuaciones químicas generales con análisis de matrices.
Palabras clave: Método matemático, matrices, balanceo de ecuaciones químicas, estabilidad.

1. Introduction

What it is a *chemical equation*? Briefly speaking, a chemical equation is only a symbolic representation of a *chemical reaction*. Actually, every chemical equation is the story of some chemical reaction. A chemical equation is not only the shorthand writing of the chemist, but it should be a mental picture of an actual reaction. To the researcher, the equation should immediately remind him as to the physical nature and properties of the reactants, *viz.*, color, state, etc., as well as the chemical result and its physical nature. Thus, a great deal of significance should be attached to the writing of chemical equations. Chemical equations play a main roll in theoretical as well as industrial chemistry. Mass balance of chemical equations as a century old problem is one of the most highly studied topics in chemical education. It always has the biggest interest for the students and the teachers as well on every level as a *magic topic*. Also, for qualitative and quantitative understanding of the chemical process estimating reactants, predicting the nature and amount of products and determining reaction conditions is necessary a balanced chemical equation. Every student which has general chemistry as a subject is bound to come across balancing chemical equations. Actually, balancing chemical equations provided an excellent demonstrative and pedagogical example of interconnection between stoichiometrical principles and linear algebra.

The substances initially involved in a chemical reaction are called *reactants*, but the newly formed substances are called the *products*. The products are new substances with properties that are different from those of reactants. Classically, chemical reactions encompass changes that strictly involve the motion of *electrons* in the forming and breaking of *chemical bonds*, although the general concept of a chemical reaction, in particular the notion of a *chemical equation*, is applicable to *transformations of elementary particles*.

In other words, a chemical equation should represent the *stoichiometry* observed in the chemical reaction. The part of chemical mathematics called *Stoichiometry* deals with the weight relations determined by chemical equations and formulas. According to it, the balancing of chemical equations is very important in this area. Since a chemical reaction, when it is feasible, is a natural process, the consequent equation is always consistent. Therefore, we must have a nontrivial solution and we should be able to obtain it assuming its existence. Such an assumption is absolutely valid and does not introduce any error. If the reaction is infeasible, then exists only a trivial solution, *i. e.*, the all coefficients are equal to zero.

2. Historical Background

The main purpose in this section is to gives a survey of selected articles on balancing chemical equations that may be useful to chemistry teachers and potential authors as background material, and to provide some comparisons of methods. The selection criteria for references were intentionally wide, in order to include a large variety of topics and former historical citations.

Balancing chemical equations in the scientific literature is considered from four points of view: mathematical, computational, chemical and pedagogical.

Now, shortly we will describe these views.

- Jones for the first time in mathematics proposed the general problem for balancing chemical equations [1]. Actually he formalized century old problem in a compact linear operator form as a Diophantine matrix equation. This problem was not solved 36 years. After that, Krishnamurthy [2] gave a mathematical method for balancing chemical equations founded by virtue of a generalized matrix inverse. He considered some elementary chemical equations, which were well known in chemistry since long time. Little bit late Das [3] offered a simple mathematical method, which was discussed in [4, 5]. A computer model for balancing some elementary chemical equations over an integer programming approach is given in [6]. Finally, in [7] by using of a reflexive *g*-inverse matrix is solved the general problem of balancing chemical equations proposed in [1]. Other mathematical results for balancing chemical equations and their stability over a nonsingular matrix method are obtained in [8]. The most general results for balancing chemical equations over a Moore-Penrose pseudoinverse matrix are obtained in [9]. In [10] is balanced a new class of chemical equations which reduces to a square $n \times n$ matrix. The solution of this class of chemical equation is founded by virtue of Drazin pseudoinverse matrix. Actually, to date in mathematics and chemistry there are only five strictly formalized consistent mathematical methods for balancing chemical equations, particularly they are methods given in [7, 8, 9, 10] and right now presented method in this work, while other so called *methods* in chemical sense have limited usage, and they are useful only for particular cases, especially for balancing chemical equations which possess atoms with integer oxidation numbers.
- There are many published articles in chemistry [11-31], which consider the use of computers to balancing chemical equations. All of these computational methods use some commercial softer packet, but unfortunately no one of them not deal with fractional oxidation numbers. Just that, it is one of their biggest weaknesses, which limit them to be applicable only in some particular cases and nothing more. It is of interest to emphasis here that same holds for the current online methods available on internet which employ only integer oxidation numbers. So, to date we do not know any computer method for balancing chemical equations to deal with fractional oxidation numbers, except previously mentioned methods of the author of this work. Actually, it was the main motive for the author to direct his research for development of new mathematical methods for balancing chemical equations in $\mathbb{Q}$ (the set of rational numbers of form p/q) in such a way to extend and generalize the current particular techniques used in chemistry right now for balancing only chemical equations in $\mathbb{N}$ (the set of natural numbers).
- University textbooks of general chemistry generally support the ion-electron technique as basic procedure for balancing chemical equation, because it makes the best use of fundamental chemical principles. Also, some authors advocated other techniques which involve less algebraic manipulation that may deserve attention – particularly in classes of chemistry and chemical engineering majors [32-58].
  Several simple chemical equations are solved by elementary algebraic techniques in [47, 59-64]. The earliest article that makes use of the linear algebra method was published by Bottomley [65]. A set of various modifications which implement this approach is documented in [33, 46, 47, 66-68]. The case when the chemical equation has no unique solution received considerable attention in the education articles [20, 69-79]. The equation represents two or more independently occurring reactions can be combined in varying stoichiometric ratios [80, 81]. Fixed ratios of reagents, observed experimentally in particular cases, are equivalent to a restriction on the coefficients that make a unique solution [20].
  It is necessary to emphasis that balancing chemical equations by *inspection* is equivalent to using the algebraic method or a computerized matrix algebra approach [82, 83]. The valence change method [32, 84-106] and the ion-electron method [83-88, 92, 101, 107-113] are also simple algebraic inspection techniques, subjected to exactly the same controls and limitations as the algebraic and matrix methods. Here it is good to emphasis that first Karslake in [114] considered balancing of ionic chemical equations. Actually, the technique suggested by García [115] can reduces the number of algebraic steps for ion-electron method. Previous both mentioned methods - the valence change method and ion-electron method begin by establishing the relative proportions of reagents taking part in separate oxidation and reduction components of a redox reaction. Then, each technique uses a lowest common multiplier to enforce a principle of conservancy - for instance, conservation of oxidation number change in the case of the oxidation number method. Johnson in his article [116] defined the equivalent term *oxidation stage change* on this subject.
  Stout in [117] presented three redox reactions as puzzles. Each one can be shown as simple redox system, which may easily be balanced using here offered method. After this article was published, the followed other debatable articles with critical accent [118-123].
- Balancing chemical equations through the pedagogical point of view is given in the articles [112, 124-131]. This approach is very interesting for the education of chemical research. A check of the hypothesis that formal reasoning and a sufficiently large mental capacity are required to balance more complex many-step equa-

tions is made over a test to determine level of intellectual development, mental capacity, and degree of field dependence/field independence of the students [131].

3. Preliminaries

Now we will introduce some well known results from the matrix algebra. Throughout, the set of $n\times n$ matrices over a field will be denoted by $\mathbb{R}^{n\times n}$.

Let $\boldsymbol{A}\in\mathbb{R}^{n\times n}$ and rank $\boldsymbol{A} = r < n$. If the matrix $\boldsymbol{A}$ has an invertible matrix of its eigenvectors, then $\boldsymbol{A}$ has an eigen decomposition. Singular value decomposition of matrix $\boldsymbol{A}$ is a factorization of the form $\boldsymbol{A} = \boldsymbol{USV}^{\mathbf{T}}$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are $n \times n$ regular matrices, $\boldsymbol{S} = \mathrm{diag}(d_1,\ldots, d_r, 0, \ldots, 0)$ and $\mathbf{T}$ denotes transpose.

The matrix

$$\boldsymbol{A}^{\boldsymbol{N}} = (\boldsymbol{V}^{\mathbf{T}})^{-1}\mathrm{diag}(1/d_1,\ldots, 1/d_r, 0,\ldots, 0)\boldsymbol{U}^{-1}$$

satisfies the equality $\boldsymbol{AA}^{\boldsymbol{N}}\boldsymbol{A} = \boldsymbol{A}$. This means that the matrix equation $\boldsymbol{AA}^{\boldsymbol{N}}\boldsymbol{A} = \boldsymbol{A}$ has at least one solution for $\boldsymbol{A}^{\boldsymbol{N}}$.

If $\boldsymbol{A}$ satisfies the identity

$$\boldsymbol{A}^r + k_1\boldsymbol{A}^{r-1} + \cdots + k_{r-1}\boldsymbol{A} = \boldsymbol{O}\ (k_{r-1} \neq 0),$$

where $\boldsymbol{O}$ is the $n \times n$ zero matrix, then the matrix

$$\boldsymbol{A}^{\boldsymbol{N}} = -\ (\boldsymbol{A}^{r-2} + k_1\boldsymbol{A}^{r-3} + \cdots + k_{r-2}\boldsymbol{I})/k_{r-1},$$

where $\boldsymbol{I}$ is the unit $n \times n$ matrix, is also a solution of the equation $\boldsymbol{AA}^{\boldsymbol{N}}\boldsymbol{A} = \boldsymbol{A}$.

Definition 3.1. *The von Neumann pseudoinverse $\boldsymbol{A}^{\boldsymbol{N}}$ of a matrix $\boldsymbol{A}\in\mathbb{R}^{n\times n}$ is the matrix which satisfies the condition*

$$\boldsymbol{AA}^{\boldsymbol{N}}\boldsymbol{A} = \boldsymbol{A}. \qquad (3.\ 1)$$

Von Neumann considered relation (3. 1) on rings of operators in [132-135].

Remark 3.2. *If $\boldsymbol{A}$ is nonsingular, then it is easily seen that $\boldsymbol{A}^{-1}$ satisfies* (3.1)*, i. e., $\boldsymbol{A}^{\boldsymbol{N}} = \boldsymbol{A}^{-1}$.*

Definition 3.3. *The characteristic equation of an $n \times n$ matrix A is the equation in one variable λ*

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0, \qquad (3.\ 2)$$

where $\det(\cdot)$ *denotes a determinant and $\boldsymbol{I}$ is an $n\times n$ identity matrix.*

Definition 3. 4. *The polynomial*

$$p(\lambda) = \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n, \quad (3.\ 3)$$

which results from evaluating the determinant (3. 2) *is the characteristic polynomial of the matrix $\boldsymbol{A}$.*

Definition 3.5. *The roots of the characteristic polynomial* (3.3) *are precisely the eigenvalues of the matrix $\boldsymbol{A}$.*

Let $\sigma(\boldsymbol{A}) = \{\lambda_i,\ 1 \le i \le n\}$ be the spectrum of $\boldsymbol{A}$.

The polynomial (3.3) of degree $n \ge 1$ with real coefficients a_v $(1 \le v \le n)$, by the fundamental theorem of algebra has n (not necessarily distinct) roots $\lambda_1, \lambda_2,\ldots, \lambda_n$.

Definition 3.6. *For any matrix $\boldsymbol{A}\in\mathbb{R}^{n\times n}$ we denote* $\mathrm{Im}\boldsymbol{A} = \{\boldsymbol{y}\in\mathbb{R}^n: \boldsymbol{y} = \boldsymbol{Ax}$ *for some $\boldsymbol{x}\in\mathbb{R}^n\}$ the image of $\boldsymbol{A}$ or range of $\boldsymbol{A}$.*

Definition 3.7. *For any matrix $\boldsymbol{A}\in\mathbb{R}^{n\times n}$ we denote* $\mathrm{Ker}\boldsymbol{A} = \{\boldsymbol{x}\in\mathbb{R}^n: \boldsymbol{Ax} = \boldsymbol{0}\}$ *the kernel of $\boldsymbol{A}$ or null space of $\boldsymbol{A}$.*

Definition 3.8. *nullity$\boldsymbol{A}$ = dim(Ker$\boldsymbol{A}$).*

Definition 3.9. *rank$\boldsymbol{A}$ = dim(Im$\boldsymbol{A}$).*

Let rank$\boldsymbol{A} = r$ and let nullity$\boldsymbol{A} = k$. According to [136], the deterministic approach is important, since it enables us to classify the chemical reaction as:

1º *infeasible* when the nullity of the reaction matrix is zero;
2º *unique* (within relative proportions) when the nullity of the reaction matrix is one;
3º *non-unique* when its nullity is bigger than one.

Possible cases of balancing chemical equations are the following:

1. If $r = n$ then $k = n - r = 0$, *i. e.*, trivial solution $\boldsymbol{x} = \boldsymbol{0}$, the reaction is infeasible.
2. If $r = n - 1$, then $k = n - r = 1$, unique solution $\boldsymbol{x} \neq \boldsymbol{0}$, *i. e.*, the reaction is feasible and is unique.
   In practical terms this means that the general procedure for obtaining these coefficients is to solve the system of linear equations derived from the principles of conservation of matter and charge, applied to the reaction element-by-element.
3. If $r < n - 1$, then $k = n - r > 1$, k (>1) linearly independent solutions $\boldsymbol{x} \neq \boldsymbol{0}$, *i. e.*, the reaction is feasible and is non-unique.

Last kind of the reactions are puzzling in that they exhibit infinite linearly independent solution all of which satisfy the chemical balance, and yet they are not all chemically feasible solutions for a given set of experimental conditions. A unique solution is obtained by imposing a chemical constraint, namely, that reactants have to react only in certain proportions.

Let $|\cdot|$ denotes a vector norm in $\mathbb{R}^n$.

Definition 3. 10. *The Lozinskiĭ measure μ on $\mathbb{R}^n$ with respect to $|\cdot|$ is defined by*

$$\mu(\boldsymbol{A}) = \lim_{\rho\to 0+}\left(\left|\boldsymbol{I} + \rho\boldsymbol{A}\right| - 1\right)/\rho. \qquad (3.4)$$

Definition 3.11. *The Lozinskiĭ measures of $\boldsymbol{A} = [a_{ij}]_{n\times n}$ with respect to the three common norms*

$$\begin{aligned} |x|_\infty &= \sup_i |x_i|, \\ |x|_1 &= \Sigma_i |x_i|, \\ |x|_2 &= (\Sigma_i |x_i|^2)^{1/2}, \end{aligned} \qquad (3.5)$$

are

$$\begin{aligned}\mu_\infty(\boldsymbol{A}) &= \sup_i(a_{ii} + \Sigma_{k,k\neq i}\,|a_{ik}|),\\ \mu_1(\boldsymbol{A}) &= \sup_k(a_{kk} + \Sigma_{i,i\neq k}\,|a_{ik}|),\\ \mu_2(\boldsymbol{A}) &= \mathrm{stab}[(\boldsymbol{A} + \boldsymbol{A}^{\mathbf{T}})/2],\end{aligned} \tag{3.6}$$

where

$$\mathrm{stab}(\boldsymbol{A}) = max\{\lambda,\ \lambda\in\sigma(\boldsymbol{A})\}$$

is the stability modulus of A.

Definition 3. 12. *The matrix* $\boldsymbol{A}$ *is stable if* $\mathrm{stab}(\boldsymbol{A}) < 0$.

4. Main Results

In this section we will give a completely new method for balancing chemical equations. Given analysis is done for arbitrary chemical equation presented in its general form.

Proposition 4. 1. *Any chemical equation may be presented in this form*

$$\sum_{j=1}^{n} x_j \prod_{i=1}^{n} \Psi_{aij}^{i} = 0, \tag{4. 1}$$

where x_j $(1 \le j \le n)$ *are unknown rational coefficients,* Ψ^i $(1 \le i \le n)$ *are chemical elements and* a_{ij} $(1 \le i, j \le n)$ *are numbers of atoms of element* Ψ^i *in j-th molecule.*

Proof. Let there exists an arbitrary chemical equation from *n* distinct elements and *n* molecules

$$\sum_{j=1}^{n} x_j \Phi_j = 0, \tag{4. 2}$$

where $\Phi_j = \Psi^1{}_{a1j}\Psi^2{}_{a2j}\cdots\Psi^n{}_{anj}$ $(1 \le j \le n)$. Then previous expression becomes

$$\sum_{j=1}^{n} x_j \Psi^1_{a1j}\Psi^2_{a2j}\cdots\Psi^n_{anj} = 0, \tag{4. 3}$$

If we write the above equation in a compact form, then immediately follows (4. 1). □

The coefficients satisfy three basic principles (corresponding to a closed input-output static model [137, 138])

- the low of conversation of atoms,
- the low of conversation of mass, and
- the time-independence of the reaction.

Theorem 4. 2. *The chemical equation* (4.1) *can be reduced to the following matrix equation*

$$\boldsymbol{Ax} = \boldsymbol{0}, \tag{4. 4}$$

where $\boldsymbol{A} = [a_{ij}]_{n\times n}$ *is a reaction matrix,* $\boldsymbol{x}^{\mathbf{T}} = (x_1, x_2, \ldots, x_n)$ *is a column vector of the coefficients* x_j $(1 \le j \le n)$ *and* $\mathbf{0}^{\mathrm{T}} = (0, 0, \ldots, 0)$ *is a null column vector of order n, and* **T** *denotes transpose.*

Proof. If we develop the molecules of the reaction (4. 1) in an explicit form, then we obtain the reaction matrix $\boldsymbol{A}$ shown below

	$\Psi^1_{a11}\Psi^2_{a21}\cdots\Psi^n_{an1}$	$\Psi^1_{a12}\Psi^2_{a22}\cdots\Psi^n_{an2}$	$\cdots$	$\Psi^1_{a1n}\Psi^2_{a2n}\cdots\Psi^n_{ann}$
Ψ^1	a_{11}	a_{12}	$\cdots$	a_{1n}
Ψ^2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
Ψ^n	a_{n1}	a_{n2}	$\cdots$	a_{nn}

From the above development we obtain that

$$\Phi_j = \sum_{i=1}^{n} a_{ij}\Psi^i \ (1 \le j \le n). \tag{4.5}$$

If we substitute (4. 5) into (4. 2), follows

$$\sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ij}\Psi^i = 0. \tag{4.6}$$

or

$$\sum_{i=1}^{n} \Psi^i \sum_{j=1}^{n} a_{ij}x_j = 0. \tag{4. 7}$$

i.e.,

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \ (1 \le i \le m). \tag{4. 8}$$

Last equation if we present in a matrix form, actually we obtain (4. 4). □

Now we will prove the following result.

Theorem 4. 3. *If* $\boldsymbol{A}^N$ *satisfies the condition* $\boldsymbol{AA}^N\boldsymbol{A} = \boldsymbol{A}$, *then*

1° $\boldsymbol{AX} = \boldsymbol{O} \Rightarrow \boldsymbol{X} = (\boldsymbol{I} - \boldsymbol{A}^N\boldsymbol{A})\boldsymbol{Q}$, ($\boldsymbol{X}$ *and* $\boldsymbol{Q}$ *are* $n\times m$ *matrices*),

2° $\boldsymbol{XA} = \boldsymbol{O} \Rightarrow \boldsymbol{X} = \boldsymbol{Q}(\boldsymbol{I} - \boldsymbol{AA}^N)$, ($\boldsymbol{X}$ *and* $\boldsymbol{Q}$ *are* $m\times n$ *matrices*),

3° $\boldsymbol{AXA} = \boldsymbol{A} \Rightarrow \boldsymbol{X} = \boldsymbol{A}^N + \boldsymbol{Q} - \boldsymbol{A}^N\boldsymbol{QAA}^N$ ($\boldsymbol{X}$ *and* $\boldsymbol{Q}$ *are* $n\times n$ *matrices*),

4° $\boldsymbol{AX} = \boldsymbol{A} \Rightarrow \boldsymbol{X} = \boldsymbol{I} + (\boldsymbol{I} - \boldsymbol{A}^N\boldsymbol{A})\boldsymbol{Q}$ ($\boldsymbol{X}$ *and* $\boldsymbol{Q}$ *are* $n\times n$ *matrices*),

5° $\boldsymbol{XA} = \boldsymbol{A} \Rightarrow \boldsymbol{X} = \boldsymbol{I} + \boldsymbol{Q}(\boldsymbol{I} - \boldsymbol{AA}^N)$ ($\boldsymbol{X}$ *and* $\boldsymbol{Q}$ *are* $n\times n$ *matrices*), *where* $\boldsymbol{Q}$ *is an arbitrary matrix.*

Proof. We will prove the theorem completely for every case.

1° Let $\boldsymbol{X} = (\boldsymbol{I} - \boldsymbol{A}^N\boldsymbol{A})\boldsymbol{Q}$. Further it follows that
$\boldsymbol{AX} = \boldsymbol{AQ} - \boldsymbol{AA}^N\boldsymbol{AQ}$, $\boldsymbol{AA}^N\boldsymbol{A} = \boldsymbol{A}$
$\Rightarrow \boldsymbol{AX} = \boldsymbol{AQ} - \boldsymbol{AQ} \Rightarrow \boldsymbol{AX} = \boldsymbol{O}$.

Conversely, assume that $\boldsymbol{AX} = \boldsymbol{O}$, then it holds that

$$(\boldsymbol{I} - \boldsymbol{A}^N\boldsymbol{A})(\boldsymbol{X} - \boldsymbol{A}^N) = \boldsymbol{X} - \boldsymbol{A}^N\boldsymbol{AX} - \boldsymbol{A}^N + \boldsymbol{A}^N\boldsymbol{AA}^N$$
$$= \boldsymbol{X} - \boldsymbol{A}^N\boldsymbol{AX} = \boldsymbol{X}.$$

Thus

$$\boldsymbol{AX} = \boldsymbol{O} \Rightarrow \boldsymbol{X} = (\boldsymbol{I} - \boldsymbol{A}^N\boldsymbol{A})\boldsymbol{Q}, \text{ for } \boldsymbol{Q} = \boldsymbol{X} - \boldsymbol{A}^N.$$

2° Now, similarly as in the previous case we will prove this part of the theorem.
Assume $\boldsymbol{X} = \boldsymbol{Q}(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}})$, then it holds that

$$\boldsymbol{X}\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{A} - \boldsymbol{Q}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A},\ \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} = \boldsymbol{A}$$
$$\Rightarrow \boldsymbol{X}\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{A} - \boldsymbol{Q}\boldsymbol{A} \Rightarrow \boldsymbol{X}\boldsymbol{A} = \boldsymbol{O}.$$

Conversely, assume that $\boldsymbol{X}\boldsymbol{A} = \boldsymbol{O}$, then it holds that

$$(\boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}})(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}) = \boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}} - \boldsymbol{X}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} + \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}$$
$$= \boldsymbol{X} - \boldsymbol{X}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} = \boldsymbol{X}.$$

Thus

$$\boldsymbol{X}\boldsymbol{A} = \boldsymbol{O} \Rightarrow \boldsymbol{X} = \boldsymbol{Q}(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}),\ \text{for}\ \boldsymbol{Q} = \boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}}.$$

3° Let $\boldsymbol{X} = \boldsymbol{A}^{\boldsymbol{N}} + \boldsymbol{Q} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{Q}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}$. Further it holds that
$\boldsymbol{A}\boldsymbol{X}\boldsymbol{A} = \boldsymbol{A}\,\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{Q}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{Q}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A},\ \boldsymbol{A}\,\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} = \boldsymbol{A}$
$\Rightarrow \boldsymbol{A}\boldsymbol{X}\boldsymbol{A} = \boldsymbol{A} + \boldsymbol{A}\boldsymbol{Q}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{Q}\boldsymbol{A} \Rightarrow \boldsymbol{A}\boldsymbol{X}\boldsymbol{A} = \boldsymbol{A}$.

Conversely, assume that $\boldsymbol{A}\boldsymbol{X}\boldsymbol{A} = \boldsymbol{A}$, then it holds that

$$\boldsymbol{B} + (\boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}}) - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}(\boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}})\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} = \boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{X}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}$$
$$+ \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} = \boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} + \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} = \boldsymbol{X}.$$

Thus

$$\boldsymbol{A}\boldsymbol{X}\boldsymbol{A} = \boldsymbol{A} \Rightarrow \boldsymbol{X} = \boldsymbol{A}^{\boldsymbol{N}} + \boldsymbol{Q} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{Q}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}},\ \text{for}\ \boldsymbol{Q} = \boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}}.$$

4° Assume $\boldsymbol{X} = \boldsymbol{I} + (\boldsymbol{I} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A})\boldsymbol{Q}$. After multiplication by $\boldsymbol{A}$, on obtains

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{A} + \boldsymbol{A}\boldsymbol{Q} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{Q},\ \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} = \boldsymbol{A}$$
$$\Rightarrow \boldsymbol{A}\boldsymbol{X} = \boldsymbol{A} + \boldsymbol{A}\boldsymbol{Q} - \boldsymbol{A}\boldsymbol{Q} \Rightarrow \boldsymbol{A}\boldsymbol{X} = \boldsymbol{A}.$$

Conversely, assume that $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{A}$, then it holds that

$$\boldsymbol{I} + (\boldsymbol{I} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A})(\boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} - \boldsymbol{I})$$
$$= \boldsymbol{I} + \boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} + \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} - \boldsymbol{I} + \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}$$
$$= \boldsymbol{I} + \boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} + \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} - \boldsymbol{I} + \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} = \boldsymbol{X}.$$

Thus

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{A} \Rightarrow \boldsymbol{X} = \boldsymbol{I} + (\boldsymbol{I} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A})\boldsymbol{Q},\ \text{for}\ \boldsymbol{Q} = \boldsymbol{X} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} - \boldsymbol{I}.$$

5° Assume that $\boldsymbol{X} = \boldsymbol{I} + \boldsymbol{Q}(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}})$, then it holds that

$$\boldsymbol{X}\boldsymbol{A} = \boldsymbol{A} + \boldsymbol{Q}\boldsymbol{A} - \boldsymbol{Q}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A},\ \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A} = \boldsymbol{A},$$
$$\Rightarrow \boldsymbol{X}\boldsymbol{A} = \boldsymbol{A} + \boldsymbol{Q}\boldsymbol{A} - \boldsymbol{Q}\boldsymbol{A} \Rightarrow \boldsymbol{A}\boldsymbol{X} = \boldsymbol{A}.$$

Conversely, assume that $\boldsymbol{X}\boldsymbol{A} = \boldsymbol{A}$, then it holds that

$$\boldsymbol{I} + (\boldsymbol{X} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} - \boldsymbol{I})(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}})$$
$$= \boldsymbol{I} + \boldsymbol{X} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} - \boldsymbol{I} - \boldsymbol{X}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} + \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} + \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}$$
$$= \boldsymbol{I} + \boldsymbol{X} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} - \boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} + \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} + \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} = \boldsymbol{X}.$$

Thus

$$\boldsymbol{X}\boldsymbol{A} = \boldsymbol{A} \Rightarrow \boldsymbol{X} = \boldsymbol{I} + \boldsymbol{Q}(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}}),\ \text{for}\ \boldsymbol{Q} = \boldsymbol{X} - \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{N}} - \boldsymbol{I}.$$ □

Remark 4. 4. *Also, the above theorem for* $\boldsymbol{A}^{\boldsymbol{N}} = \boldsymbol{B}$ *was employed for solving of lot of cyclic linear complex vector functional equations* [139-141].

If $\boldsymbol{X}$ is an $n\times 1$ matrix, according to Theorem 4.3 the solution of the homogeneous system of equations

$$\boldsymbol{A}\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = 0$$

obtains this form

$$\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = (\boldsymbol{I} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A})\begin{bmatrix} q_1 \\ q_2 \\ . \\ . \\ . \\ q_n \end{bmatrix}, \qquad (4.\ 9)$$

where $q_1, \ldots, q_n$ are arbitrary.

Definition 4.5. *Chemical equation* (4.1) *is stable if* $\text{stab}(\boldsymbol{A}) < 0$.

Lemma 4.6. *For any nonsingular matrix* $\boldsymbol{U}$ *and any vector norm* $|\cdot|$, *with the induced Lozinskiĭ measure* μ, $|\boldsymbol{U}\boldsymbol{x}|$ *defines another vector norm and its induced matrix measure* $\mu_{\boldsymbol{U}}$ *is given by*

$$\mu_{\boldsymbol{U}}(\boldsymbol{A}) = \mu(\boldsymbol{U}\boldsymbol{A}\boldsymbol{U}^{-1}). \qquad (4.\ 10)$$

Proof. The proof of this lemma follows directly from the Definition 3. 10. □

Theorem 4.7. *For any matrix* $\boldsymbol{A}\in\mathbb{R}^{n\times n}$ *it holds*

$$\text{stab}(\boldsymbol{A}) = \inf\{\mu(\boldsymbol{A}),\ \mu\ \textit{is a Lozinskiĭ measure on}\ \mathbb{R}^n\}. \qquad (4.11)$$

Proof. The relation (4.11) obviously holds for diagonalizable matrices in view of (4.10) and the first two relations in (3.6). Furthermore, the infimum in (4.11) can be achieved if the matrix $\boldsymbol{A}$ is diagonalizable. The general case can be shown based on this observation, the fact that $\boldsymbol{A}$ can be approximated by diagonalizable matrices in $\mathbb{R}$ and the continuity of $\mu(\cdot)$, which is implied by the property

$$|\mu(\mathfrak{S}) - \mu(\aleph)| \le |\mathfrak{S} - \aleph|.$$ □

Remark 4.8. *From the above proof it follows that*

$$\text{stab}(\boldsymbol{A}) = \inf\{\mu_\infty(\boldsymbol{U}\boldsymbol{A}\boldsymbol{U}^{-1}),\ \det\boldsymbol{U} \neq 0\}.$$

The same relation holds if μ_∞ is replaced by μ_1.

Corollary 4.9. *Let* $\boldsymbol{A}\in\mathbb{R}$. *Then* $\text{stab}(\boldsymbol{A}) < 0 \Leftrightarrow \mu(\boldsymbol{A}) < 0$ *for some Lozinskiĭ measure* μ *on* $\mathbb{R}^n$.

More results for stability criteria are obtained in works [142, 143].

5. An Application of the Main Results

In this section will be applied above method on many chemical equations for their balancing. All chemical equations balanced here appear first time in professional literature and they are chosen with an intention to be avoided to date all well known chemical equations which were repeated many times in the chemical journals for explanation of certain particular techniques for balancing of some chemical equations using only atoms with integer oxidation numbers.

1º First we will consider an infeasible reaction, *i. e.*, the case when the nullity of the reaction matrix is zero.

Example 5. 1. Consider chemical equation

$$x_1\,Fe_2(SO_4)_3 + x_2\,PrTlTe_3 + x_3\,H_3PO_4 \qquad (5.\ 1)$$
$$= x_4\,Fe(H_2PO_4)_2{\cdot}H_2O + x_5\,Pr_2(SO_4)_3 + x_6\,Tl_{1.99}(SO_3)_3$$
$$+ x_7\,Te_2O_3 + x_8\,H_2O.$$

The reaction matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 0 & -1 & 0.00 & 0.00 & 0 & 0 \\ 3 & 0 & 0 & 0 & -3.00 & -3.00 & 0 & 0 \\ 12 & 0 & 4 & -9 & -12.0 & -9.00 & -3 & -1 \\ 0 & 1 & 0 & 0 & -2.00 & 0.00 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.00 & -1.99 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0.00 & 0.00 & -2 & 0 \\ 0 & 0 & 3 & -6 & 0.00 & 0.00 & 0 & -2 \\ 0 & 0 & 1 & -2 & 0.00 & 0.00 & 0 & 0 \end{bmatrix}.$$

is obtained from this scheme

	$Fe_2(SO_4)_3$	$PrTlTe_3$	H_3PO_4	$Fe(H_2PO_4)_2{\cdot}H_2O$	$Pr_2(SO_4)_3$	$Tl_{1.99}(SO_3)_3$	Te_2O_3	H_2O
Fe	2	0	0	-1	0	0	0	0
S	3	0	0	0	-3	-3	0	0
O	12	0	4	-9	-12	-9	-3	-1
Pr	0	1	0	0	-2	0	0	0
Tl	0	1	0	0	0	-1.99	0	0
Te	0	3	0	0	0	0	-2	0
H	0	0	3	-6	0	0	0	-2
P	0	0	1	-2	0	0	0	0

The rank of the above matrix is $r = 8$. Since the nullity of the reaction matrix is $k = n - r = 8 - 8 = 0$, then we have only a trivial solution $\boldsymbol{x} = \boldsymbol{0}$, that means that the reaction is infeasible.

2º Next, we will consider the case when the chemical reaction is feasible and is unique, *i. e.*, the nullity of its reaction matrix is one. Here we will balance many special chemical equations with a goal to show the power of the offered mathematical method.

Example 5. 2. Consider this equation

$$x_1[4Yb(CN)_3{\cdot}3Yb(CN)_2] + x_2CsRu(CN)_2F_2 \qquad (5.\ 2)$$
$$+ x_3CsRu(CN)_4 + x_4CsHF_2 + x_5[PtF_3{\cdot}7H_2O]$$
$$= x_6[Pt(NH_3)_2(C_5H_4ON)]_2(NO_3)_2{\cdot}2H_2O$$
$$+ x_7Cs_{3.99}Yb(CN)_6 + x_8HRuF_{2.97} + x_9NO_2.$$

From the scheme given below

	$[4Yb(CN)_3{\cdot}3Yb(CN)_2]$	$CsRu(CN)_2F_2$	$CsRu(CN)_4$	$CsHF_2$	$[PtF_3{\cdot}7H_2O]$	$[Pt(NH_3)_2(C_5H_4ON)]_2(NO_3)_2{\cdot}2H_2O$	$Cs_{3.99}Yb(CN)_6$	$HRuF_{2.97}$	NO_2
Yb	7	0	0	0	0	0	-1	0	0
C	18	2	4	0	0	-10	-6	0	0
N	18	2	4	0	0	-8	-6	0	-1
Cs	0	1	1	1	0	0	-3.99	0	0
Ru	0	1	1	0	0	0	0	-1	0
F	0	2	0	2	3	0	0	-2.97	0
H	0	0	0	1	14	-24	0	-1	0
Pt	0	0	0	0	1	-2	0	0	0
O	0	0	0	0	7	-10	0	0	-2

is derived the reaction matrix

$$\boldsymbol{A} = \begin{bmatrix} 7.00 & 0 & 0 & 0 & 0.00 & 0.00 & -1.00 & 0.00 & 0 \\ 18.0 & 2 & 4 & 0 & 0.00 & -10.0 & -6.00 & 0.00 & 0 \\ 18.0 & 2 & 4 & 0 & 0.00 & -8.00 & -6.00 & 0.00 & -1 \\ 0.00 & 1 & 1 & 1 & 0.00 & 0.00 & -3.99 & 0.00 & 0 \\ 0.00 & 1 & 1 & 0 & 0.00 & 0.00 & 0.00 & -1.00 & 0 \\ 0.00 & 2 & 0 & 2 & 3.00 & 0.00 & 0.00 & -2.97 & 0 \\ 0.00 & 0 & 0 & 1 & 14.0 & -24.0 & 0.00 & -1.00 & 0 \\ 0.00 & 0 & 0 & 0 & 1.00 & -2.00 & 0.00 & 0.00 & 0 \\ 0.00 & 0 & 0 & 0 & 7.00 & -10.0 & 0.00 & 0.00 & -2 \end{bmatrix}.$$

The rank of the above matrix is $r = 8$. Since the nullity of the reaction matrix is $k = n - r = 9 - 8 = 1$, then we have a non-trivial solution $\boldsymbol{x} \neq \boldsymbol{0}$, that means that the reaction is feasible.

Singular value decomposition of the matrix $\boldsymbol{A}$ is given by the expression $\boldsymbol{A} = \boldsymbol{USV}^{\mathbf{T}}$, where

$$\boldsymbol{U} = \begin{bmatrix} -0.108818459557726 & -0.525441407603166 \\ 0.214701170957876 & 0.450899256172441 \\ 0.016066764162855 & 0.057213059874600 \\ -0.484373182392954 & 0.098454545415393 \\ 0.345438356921560 & -0.304312644967055 \\ 0.547145909762408 & 0.015307612109590 \\ -0.523852079829627 & 0.148809263996733 \\ -0.118613333029776 & -0.568932626114908 \\ 0.000000000000000 & 0.262612865719445 \end{bmatrix}$$

$$\begin{bmatrix} -0.483709300771011 & -0.025011484625246 \\ 0.483572372868432 & 0.046136078870445 \\ -0.046373294704348 & -0.453189307512911 \\ -0.004184833431819 & 0.765218064375209 \\ 0.180768863041104 & 0.278205209655966 \\ -0.216740730980893 & 0.303122641025548 \\ 0.054658267642886 & -0.187510779304511 \\ 0.615148849855887 & -0.043184335536401 \\ -0.262612865719445 & 0.000000000000000 \end{bmatrix}$$

$$\begin{bmatrix} -0.005419903080133 & -0.031804241854372 \\ 0.007672804708575 & -0.054870209881318 \\ -0.215770782653738 & -0.859989638320084 \\ 0.006660853893509 & -0.406023012474453 \\ -0.310494489641576 & -0.099401772162723 \\ -0.531515502920401 & 0.004639387101885 \\ -0.755862036723293 & 0.282361389851866 \\ -0.055383920003139 & -0.043711496085823 \\ 0.000000000000000 & 0.000000000000000 \end{bmatrix}$$

$$\begin{bmatrix} -0.631375254382829 & -0.051094299859864 \\ -0.651011866763206 & -0.051017560379749 \\ 0.042288273104927 & 0.030745308960465 \\ 0.021835681349140 & 0.019913511012403 \\ -0.253995146388407 & -0.036993147062515 \\ 0.252752992636479 & 0.000040477189790 \\ -0.101377108856827 & 0.034418504872053 \\ 0.191334300606536 & -0.382183260829536 \\ 0.000000000000000 & -0.919145030018058 \end{bmatrix}$$

$$\begin{bmatrix} -0.274195885354736 \\ -0.291776689266259 \\ 0.008044453565360 \\ -0.065884180607602 \\ 0.711210986578713 \\ -0.463813345852432 \\ 0.052627541396678 \\ -0.307119873865167 \\ 0.131306432859722 \end{bmatrix},$$

$\boldsymbol{S}$ = diag(35.7077953660721, 25.0075595691818, 5.14897940671986, 4.2433996813797, 2.45040387879316, 1.90114742807227, 0.398185956988948, 0.143150036571975, 0)

and

$$\boldsymbol{V}^{\mathbf{T}} = \begin{bmatrix} -0.530036698530604 & 0.732714340587708 \\ -0.059156306529345 & 0.072498546564787 \\ -0.113897656786349 & 0.151621968119400 \\ -0.020163530549436 & -0.028576007414021 \\ -0.305399190077898 & -0.454751736894276 \\ 0.759532806845423 & 0.410536471688309 \\ 0.175410396224260 & -0.240151938188209 \\ 0.020478826773642 & 0.032242393871809 \\ 0.028904082733182 & 0.003998031290798 \end{bmatrix}$$

$$\begin{bmatrix} 0.059736716329278 & -0.399151055338640 \\ -0.459753215050118 & 0.034965435364956 \\ -0.121500006131151 & 0.270763503905774 \\ -0.413845180321981 & -0.005889683061020 \\ -0.369175414664211 & -0.319000177550112 \\ -0.263741188677449 & -0.201750581917647 \\ 0.335429576497838 & -0.738668379535351 \\ 0.529746095274656 & 0.277464273981295 \\ 0.005881629189292 & 0.032038743661973 \end{bmatrix}$$

$$\begin{bmatrix} 0.079285062134118 & 0.107422089227711 \\ -0.195143499527296 & -0.327160490685546 \\ -0.214848014339865 & -0.543947997988063 \\ -0.071250899726732 & 0.297270164070787 \\ 0.443741704162715 & 0.160853971619241 \\ 0.267194927606734 & 0.080388509256987 \\ -0.291468054745698 & -0.288250414531082 \\ 0.350845387894457 & 0.139380842684048 \\ -0.654255753540559 & 0.601935129168877 \end{bmatrix}$$

$$\begin{bmatrix} -0.011424283590129 & 0.011167975695062 \\ 0.071034717469835 & -0.653571612482181 \\ -0.325231684333164 & 0.602840498618986 \\ 0.692728828581090 & 0.424219052629831 \\ -0.425389335857176 & 0.108441064494780 \\ -0.219547150578892 & 0.081575306129862 \\ 0.128620556596709 & 0.095155332826569 \\ 0.046776681580951 & -0.042802903689246 \\ -0.401604696573164 & -0.006350694329692 \end{bmatrix}$$

$$\begin{bmatrix} 0.034926906301773 \\ 0.448700496904865 \\ 0.254423990564171 \\ 0.272384005539498 \\ 0.215370240964770 \\ 0.107685120482385 \\ 0.244488344112414 \\ 0.703124487469035 \\ 0.215370240964769 \end{bmatrix}.$$

In linear algebra, the singular value decomposition is an important factorization of rectangular real and complex matrices, with several applications in applied sciences. Actually, the singular value decomposition can be seen as a generalization

of the spectral theorem to arbitrary, not necessarily square matrices. Since, the matrix $\boldsymbol{A}$ has a matrix of eigenvectors that is not invertible, *i. e.*, the matrix $\boldsymbol{A}$ does not have an eigen decomposition, then $\boldsymbol{A}$ can be presented by its singular value decomposition. The von Neumann pseudoinverse $\boldsymbol{A^N}$ of the matrix $\boldsymbol{A}$ determined by the formula $\boldsymbol{A^N} = (\boldsymbol{V^T})^{-1}\mathrm{diag}(1/d_1,\ldots, 1/d_8, 0)\boldsymbol{U}^{-1}$ is

$$\boldsymbol{A^N} = \begin{bmatrix}
0.101523256567670 & 0.321803887120054 \\
-0.045223147018723 & 2.643569604152050 \\
0.061229664532195 & -2.769586211377240 \\
-0.047374220591932 & 0.136340300180892 \\
-0.025538092341311 & 0.253565957729192 \\
0.012613778846674 & 0.330397410308811 \\
0.012633132259008 & -0.919353959433015 \\
-0.034251335322107 & 1.751346157688120 \\
-0.026853724152914 & 1.433111472758320
\end{bmatrix}$$

$$\begin{bmatrix}
-0.288108429146632 & -1.188150902038370 \\
-2.485818145278540 & -1.420833955106290 \\
2.597366632008940 & 1.882364109752350 \\
-0.079944559167976 & -0.379557722145593 \\
0.569019045271805 & -1.535863922093850 \\
-0.413165815277642 & 0.435071248156406 \\
0.836975164068040 & 0.435222179359037 \\
-1.633950741785980 & -1.074035211448490 \\
-1.271285637926960 & -0.911850350422177
\end{bmatrix}$$

$$\begin{bmatrix}
0.610923180005891 & 0.305461590002947 \\
-0.658983155079063 & -0.450181232711944 \\
0.420984017508641 & 0.331181663926733 \\
0.217976731579250 & 0.108988365789626 \\
0.679224462547416 & 0.339612231273709 \\
-0.258919826562618 & -0.129459913281308 \\
0.241199512637992 & 0.120599756318995 \\
-0.335319169083302 & -0.245245791438202 \\
-0.187299838407709 & -0.153994746790060
\end{bmatrix}$$

$$\begin{bmatrix}
-0.289337204026299 & 0.033695457973420 \\
-0.316562029131021 & 0.157751458873511 \\
0.428607651725322 & -0.172219579368304 \\
-0.331619544143528 & 0.056395741012914 \\
-0.178766646389184 & -0.177414996999005 \\
0.088296451926710 & -0.082768404968829 \\
0.088431925813047 & -0.082378795364977 \\
-0.239759347254723 & 0.117395415902143 \\
-0.187976069070377 & 0.161825834831369
\end{bmatrix}$$

$$\begin{bmatrix}
0.610923180005892 \\
-0.038293499906653 \\
-0.199705637663771 \\
0.217976731579252 \\
0.679224462547419 \\
-0.258919826562618 \\
0.241199512637993 \\
-0.007732962186752 \\
-0.376955010821503
\end{bmatrix}.$$

Required coefficients of the chemical equation (5. 2), according to the formula (4. 9) are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = (\boldsymbol{I} - \boldsymbol{A^N A}) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.087194107660614 \\ 1.120169049513394 \\ 0.635162835008073 \\ 0.679999542439780 \\ 0.537666171040854 \\ 0.268833085520413 \\ 0.610358753624375 \\ 1.755331884521518 \\ 0.537666171040864 \end{bmatrix}.$$

Now balanced chemical equation (5. 2) obtains this form

$$\begin{gathered}
0.087194107660614[4Yb(CN)_3\cdot 3Yb(CN)_2] \\
+ 1.120169049513394CsRu(CN)_2F_2 \\
+ 0.635162835008073CsRu(CN)_4 \\
+ 0.679999542439780CsHF_2 \\
+ 0.537666171040854[PtF_3\cdot 7H_2O] \\
= 0.268833085520413[Pt(NH_3)_2(C_5H_4ON)]_2(NO_3)_2\cdot 2H_2O \\
+ 0.610358753624375Cs_{3.99}Yb(CN)_6 \\
+ 1.755331884521518HRuF_{2.97} \\
+ 0.537666171040864NO_2.
\end{gathered}$$

If we multiply above equality by 454159.1291252759 we obtain the equality in its conventional form

$$\begin{gathered}
39600[4Yb(CN)_3\cdot 3Yb(CN)_2] + 508735CsRu(CN)_2F_2 \\
+ 288465CsRu(CN)_4 + 308828CsHF_2 \\
+ 244186[PtF_3\cdot 7H_2O] \\
= 122093[Pt(NH_3)_2(C_5H_4ON)]_2(NO_3)_2\cdot 2H_2O \\
+ 277200Cs_{3.99}Yb(CN)_6 + 797200HRuF_{2.97} + 244186NO_2.
\end{gathered}$$

The eigenvalues of the matrix $(\boldsymbol{A} + \boldsymbol{A^T})/2$ are

$$\begin{gathered}
\lambda_1 = -18.060777308031500,\ \lambda_2 = 20.041652979700600, \\
\lambda_3 = 14.665879176680500,\ \lambda_4 = -5.617846328393410, \\
\lambda_5 = 2.319213465566350,\ \lambda_6 = -1.937661578812980, \\
\lambda_7 = 1.071305345298540,\ \lambda_8 = -0.696805962104278, \\
\lambda_9 = 0.215040210096249.
\end{gathered}$$

The Lozinskii measures of $\boldsymbol{A}$ given by (3. 12) with respect to the three common norms (3. 11) are

$$\begin{gathered}
\mu_\infty(\boldsymbol{A}) = \max\,(8, 40, 39, 6.99, 3, 9.97, 40, 3, 15) = 40, \\
\mu_1(\boldsymbol{A}) = \max\,(43, 8, 10, 4, 25, 54, 16.99, 4.97, -1) = 54, \\
\mu_2(\boldsymbol{A}) = \lambda_2 = 20.041652979700600.
\end{gathered}$$

Since $\mu_2(\boldsymbol{A}) > 0$ and definition 4. 5 immediately follows that the chemical equation (5. 2) is unstable.

With this method we balanced successfully lot of chemical equations and some of them are given below as examples. The research shown that considered chemical equations are unstable too.

Example 5.3.

$$171400AuO\cdot[Pt(C_{10}H_8N_2)_3]F_2\cdot 6H_2O + 5592972W_4Fe(CN)_6 + 1198494C_4H_3OsNa_2OS_7 + 2460568H_2CO_3 + 10672397Os_2O + 10157544NO_2 + 1198086TeO_3 = 171400AuPtOsTe_{6.99} + 2796486Fe_2(SO_4)_3 + 22371888WOs(CN)_2 + 1210600Na_{1.98}CO_3 + 342800HF + 7172109H_2O. \quad (5.3)$$

Example 5.4.

$$153936AgAuPtSe_6\cdot[(NH_3)HClO_3] + 132534C_4H_3AuCs_2OS_7 + 618492Li_4Mn(CN)_6 + 70004Ru_3(CO)_{12} + 1093749Au_2O_3 + 1390920HNO_3 = 153936Pt(NH_3)\cdot ClNO_3 + 133200Cs_{1.99}CO_3 + 2473968LiAu(CN)_2 + 309246Mn_2(SO_4)_3 + 210012RuO_2 + 923616SeO_3 + 153936AgO + 971229H_2O. \quad (5.4)$$

Example 5.5.

$$79910NH_4ClO_4 + 140000NaY(OH)_4 + 153330Ru(SCN)_3 + 55916PBr_5 + 29905TiCl_2\cdot CrI_4 + 69860BeCO_3 + 69860Rb_2ZrO_3 + 269950ZnAt_2 + 10050CAt_2I_2 = 140000Rb_{0.998}YAt_4 + 153330RuS_2 + 69860BeZrO_3 + 269950Zn(CN)_2 + 140000NaHBr_{1.997} + 55916H_3PO_4 + 29905TiCrO_4 + 139720ClI + 153330H_2SO_4 + 132616H_2O. \quad (5.5)$$

Example 5.6.

$$207C_{55}H_{72}SrN_4O_5 + 10[WCl_4(NTeCl)]_2 + 4347HNO_3 + 3338RuPO_4 + 1503SrCl_2 + 2095(NH_4)_3[PO_4\cdot 12MoO_3] + 1438PdCl_4 + 20NaLiCl_2\cdot K_4Co(CN)_6 + 10Li_2Pb_2O_3 + 15CrI_3 + 15BeSiO_3 = 15BeCO_3 + 15SiCrKO_4 + 45KI(CN)_2 + 25140MoO_3 + 1900RuSr_{0.9}(CN)_6 + 1438RuPdCl_6 + 5433H_3PO_4 + 20PbKWCoCl_{11} + 5Na_4Li_8N_{22}Se_4Cl_{10} + 14046H_2O. \quad (5.6)$$

Example 5.7.

$$2448C_{55}H_{72}BaN_4O_5 + 1188[WF_4(NTeF)]_2 + 52605HNO_3 + 34572PtPO_4 + 20292BaF_2 + 23085(NH_4)_3[PO_4\cdot 12MoO_3] + 11832PdF_4 + 2400\ AgCsF_2\cdot K_4Mn(CN)_6 + 1200Pb_2O_3 + 1800CrI_3 + 1800BeSiO_3 + 800AcAt_3 = 800Ac + 1800BeCO_3 + 1800SiCrKO_4 + 5400KI(CN)_2 + 277020MoO_3 + 22740PtBa(CN)_6 + 11832PtPdF_6 + 57657H_3PO_4 + 2400AtCsPbKW_{0.99}AgMnF_{14} + 594N_2Te_4 + 166455H_2O. \quad (5.7)$$

Example 5.8.

$$49392C_{44}H_{34}BrN_2O_2P_2 + 15144C_{44}H_{32}F_5NP_2Pt + 2904CsLiGaHfFTe_3 + 35637(C_{22}H_{22}O_5)_2 + 924CuCoO_4 + 1848Sb_2N_2PS_7 + 57888Au_2O + 57888MnO_2 + 16464TiFeCl_6 + 50400GeO_2 + 7572Os_2O_3 + 924Y_2O_3\cdot HgS + 76776Sr(CeI_4)_2 = 924C_{69}H_{39}Cl_6CuN_{27}O_{19}S_7 + 57888C_{44}H_{34}Au_2MnO_3P_2 + 16800C_{44}H_{44}O_3Ge_3Ti_{0.98} + 16464C_{44}H_{34}FeBr_3ClO_5 + 7572(C_{22}H_{16}O_4OsPPt)_2 + 462Sb_8Co_2Y_4N_2O_{23} + 76776SrClF + 76776(CeI_4)_2 + 924HgF_2 + 2904CsLiO_2 + 1452Ga_2O + 2904HfO_2 + 4400Te_{1.98}O_3 + 91752HNO_3 + 47352H_2O + 7392SO_2. \quad (5.8)$$

Example 5.9.

$$74744(NH_3)_3[(PO)_4\cdot 12MoO_3] + 530808HoHgTlZrS_6 + 2121414In_3SrBkCl_{12} + 1061640AgRuAuOs_8 + 713874C_4H_3AuLi_2OS_7 + 24364584KAu(CN)_2 + 3045573MgMn_2(SO_4)_4 + 530820PbCrO_4 + 2069214Sn_3(PtO_4)_3 + 6207642BeSiO_3 + 6207642CuCsCl_3 + 1060707N_2SiSe_6 + 22761354YbAlF_5 + 2069214AcAt_3 + 3182121Te_2O + 16811412H_2CO_3 + 8957322HClO = 6207642YbBePtSAtCsF_{13} + 1061640[Ru(C_{10}H_8N_2)_3]Cl_2\cdot 6H_2O + 3182121[TeCl_4(NSeInCl_3)]_2 + 896928(NH_4)_2MoO_4 + 6091146K_4Mn(CN)_6 + 2121414SrO\cdot BkCl_3 + 265410Li_2Cr_2O_7 + 3045573MgS_2O_3 + 530808HoTlS_3 + 298976Li_3PO_4 + 530820Ag_2PbO_2 + 6207642SnSO_4 + 16553712YbF_2 + 265404Hg_2S + 530808ZrO_2 + 3103821Cu_2O + 11380677Al_2O_3 + 1089060Ac_{1.9}O_3 + 7268349SiO_2 + 26140098AuO + 8493120OsO_3. \quad (5.9)$$

Example 5.10.

$$7731000CaBeSbSAtCsF_{13} + 1502160[Ru(C_{10}H_8N_2)_3]Cl_2\cdot 6H_2O + 9273600[PtCl_4(NTeInCl_3)]_2 + 12369600Ca(GaH_2S_4)_2 + 1560720(NH_4)_2MoO_4 + 12054510K_4Yb(CN)_6 + 375540Na_2Cr_2O_7 + 6027255MgS_2O_3 + 37709196LaTlS_3 + 520240Na_3PO_4 + 751080Ag_2PbO_2 + 7731000SnSO_4 + 24739200HoHS_4 + 6182400CeCl_3 + 37709196HfO_2 + 3865500Cu_2O + 9748791W_2O_3 + 1288500Am_2O_3 + 10822200SiO_2 + 25438050Au_2O + 12017280TeO_3 + 6182400CdO + 18854598Hg_2S = 130060(NH_3)_3[(PO)_4\cdot 12MoO_3] + 37709196LaHgTlHfS_6 + 6182400In_3CdCeCl_{12} + 1502160AgRuAuTe_8 + 1155900C_4H_3AuNa_2OS_7 + 48218040KAu(CN)_2 + 6027255MgYb_2(SO_4)_4 + 2577000Sn_3(SbO_4)_3\cdot AmAt_3 + 7731000CuCsCl_3 + 24739200GaHoH_2S_4 + 3091200N_2SiTe_6 + 20100600CaW_{0.97}F_5 + 751080PbCrO_4 + 16332180H_2CO_3 + 7731000BeSiO_3 + 54000120HClO + 9273600Pt_2O. \quad (5.10)$$

3º Now, we will consider the case when the chemical reaction is non-unique, *i. e.*, when the nullity of its reaction matrix is bigger than one. For this purpose, additionally we will solve more one chemical equation.

Example 5. 11. As a special case of this section we will balance this chemical equation

$$x_1NH_3OsO_2 + x_2NHOsO{\cdot}H_2O \quad (5.11)$$
$$= x_3HOs{\cdot}NO{\cdot}H_2O + x_4NH_4OsO_{2.99}.$$

For that purpose we will use the above von Neumann pseudoinverse matrix method for balancing chemical equations.

The reaction matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -1 & -1.00 \\ 3 & 3 & -3 & -4.00 \\ 1 & 1 & -1 & -1.00 \\ 2 & 2 & -2 & -2.99 \end{bmatrix},$$

follows from the scheme given below

	NH_3OsO_2	$NHOsO{\cdot}H_2O$	$HOs{\cdot}NO{\cdot}H_2O$	$NH_4OsO_{2.99}$
N	1	1	-1	-1
H	3	3	-3	-4
Os	1	1	-1	-1
O	2	2	-2	-2.99

The rank of the above matrix is $r = 2$. Since the nullity of the reaction matrix is $k = n - r = 4 - 2 = 2 > 1$, then we have infinite number of linearly independent solutions $\boldsymbol{x} \neq \boldsymbol{0}$. Here we will determine the general solution of (5. 11) as well as its minimal solution. First we will determine its general solution. From the above chemical equation follows this system of linear equations

$$x_1 + x_2 = x_3 + x_4,$$
$$3x_1 + 3x_2 = 3x_3 + 4x_4,$$
$$x_1 + x_2 = x_3 + x_4,$$
$$2x_1 + 2x_2 = 2x_3 + 2.99x_4.$$

The general solution of this system is

$$x_4 = 0, x_3 = x_1 + x_2,$$

where x_1 and x_2 are arbitrary real numbers.

Now, the balanced equation has a form

$$x_1NH_3OsO_2 + x_2NHOsO{\cdot}H_2O = (x_1 + x_2)HOs{\cdot}NO{\cdot}H_2O,$$

where x_1 and x_2 are arbitrary real numbers.

This is okay from a mathematical view point. It means that the reaction (5.11) has infinity number modifications, but in chemistry it is important to be determined unique minimal coefficients $x_1, x_2 \in \mathbb{R}$.

The singular value decomposition of the matrix $\boldsymbol{A}$ is $\boldsymbol{A} = \boldsymbol{USV}^{\mathbf{T}}$, where

$$\boldsymbol{U} = \begin{bmatrix} -0.234014498884303 & 0.590972553257766 \\ -0.774225777939954 & 0.024968247278484 \\ -0.234014498884303 & 0.590972553257767 \\ -0.539489456242780 & -0.548524811854334 \end{bmatrix}$$

$$\begin{bmatrix} 0.620986215268903 & -0.458655400114213 \\ -0.548295143453770 & -0.315156258173821 \\ -0.083767741379856 & 0.767444865193612 \\ 0.553833478236131 & 0.318339654721031 \end{bmatrix},$$

$\boldsymbol{S}$ = diag(8.469659585212530, 0.452732272560185, 0, 0) and

$$\boldsymbol{V}^{\mathbf{T}} = \begin{bmatrix} -0.456887930989605 & -0.456887930989605 \\ 0.352968485632034 & 0.352968485632034 \\ -0.815775583212880 & 0.437597190376265 \\ -0.034305458754441 & -0.689329649472200 \end{bmatrix}$$

$$\begin{bmatrix} 0.456887930989605 & 0.611359350585328 \\ -0.352968485632035 & 0.791353109839019 \\ -0.378178392836615 & 0.000000000000000 \\ -0.723635108226640 & 0.000000000000000 \end{bmatrix}.$$

Analogously as in the previously section, we will determine the von Neumann pseudoinverse $\boldsymbol{A}^{\boldsymbol{N}}$ of the matrix $\boldsymbol{A}$. It is given by the formula $\boldsymbol{A}^{\boldsymbol{N}} = (\boldsymbol{V}^{\mathbf{T}})^{-1}\text{diag}(1/d_1, 1/d_2, 0, 0)\boldsymbol{U}^{-1}$, *i.e.*,

$$\boldsymbol{A}^{\boldsymbol{N}} = \begin{bmatrix} 0.473369923758612 & 0.061231152190324 \\ 0.473369923758610 & 0.061231152190324 \\ -0.473369923758612 & -0.061231152190324 \\ 1.016098426883760 & -0.012242149721491 \end{bmatrix}$$

$$\begin{bmatrix} 0.473369923758614 & -0.398549985377433 \\ 0.473369923758612 & -0.398549985377431 \\ -0.473369923758614 & 0.398549985377433 \\ 1.016098426883770 & -0.997735202301526 \end{bmatrix}.$$

Now, immediately on can determine required coefficient of the chemical equation (5. 11). According to the formula (4. 9) their values are explicitly given by the following elementary matrix expression

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = (\boldsymbol{I} - \boldsymbol{A}^{\boldsymbol{N}}\boldsymbol{A}) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.66666666666666533 \\ 0.66666666666666731 \\ 1.33333333333333467 \\ 0.00000000000000000 \end{bmatrix}.$$

Now balanced chemical equation (5. 11) obtains this form

$$0.66666666666666533NH_3OsO_2$$
$$+ 0.66666666666666731NHOsO{\cdot}H_2O$$
$$= 1.33333333333333467HOs{\cdot}NO{\cdot}H_2O,$$

i.e.,

$$2/3\ NH_3OsO_2 + 2/3\ NHOsO{\cdot}H_2O = 4/3HOs{\cdot}NO{\cdot}H_2O.$$

If we multiply above equality by 3/2 we obtain the equality in its convntional form

$$NH_3OsO_2 + NHOsO{\cdot}H_2O = 2HOs{\cdot}NO{\cdot}H_2O.$$

The eigenvalues of the matrix $(\boldsymbol{A} + \boldsymbol{A}^{\mathbf{T}})/2$ are

$\lambda_1 = 4.402163583725810$, $\lambda_2 = -3.477656019318620$,
$\lambda_3 = 0.066579373837709$, $\lambda_4 = -\ 0.981086938244899$.

The Lozinskiĭ measures of $\boldsymbol{A}$ given by (3. 12) with respect to the three common norms (3. 11) are

$$\mu_\infty(\boldsymbol{A}) = \max\,(4, 13, 2, 4.99) = 13,$$
$$\mu_1(\boldsymbol{A}) = \max\,(7, 7, 5, 4.99) = 7,$$
$$\mu_2(\boldsymbol{A}) = \lambda_1 = 4.402163583725810.$$

Since $\mu_2(\boldsymbol{A}) > 0$ and definition 4. 5 immediately follows that the chemical equation (5. 11) is unstable.

Similar classes of chemical equations are considered in [26, 44, 144, 145], but unfortunately as unsolved problems. The solutions of these equations are obtained in [10] from the same author of this work.

Remark 5. 12. *This work and previously published works* [7-10] *make a circled scientific whole. Actually, by these works is completely solved century old problem of balancing chemical equations in its general form by using of generalized matrix inverses. Accurately speaking, it means that the general problem of balancing chemical equations from now remains behind us only like a history.*

6. Conclusion

The practical superiority of the matrix procedure as the most general tool for balancing chemical equations is demonstrable. By this method are balanced completely new classes of chemical equations with atoms which possess fractional oxidation numbers. Obtained results shown that employed singular matrix method founded by virtue of the von Neumann pseudoinverse matrix works perfectly for the chemical equations presented as a square matrix equation.

Here presented method is unique method both in mathematics and chemistry which balances chemical equations with atoms which possess fractional as well as integer oxidation numbers, while all to date known methods for balancing chemical equations give an opportunity to balance chemical equations only with atoms which possess integer oxidation numbers. This is the main advantage of the method in relation of other known methods.

In other words, the mathematical method given here is applicable for all possible cases for balancing chemical equations, does not matter what kind of atoms they possess - fractional or integer oxidation numbers.

For all considered chemical equations is made a stability analysis, and as shown results all of them are unstable. This stability analysis is founded by virtue of the Lozinskiĭ measures of the reaction matrix.

Here developed method for balancing chemical equations gives a perfect opportunity for an application of group theory for determination of all Sylow *p*-subgroups of permutation of the coefficients of the balanced chemical equation.

Also, the author of this work wants to emphasis here, that by this work and the others previous published matrix methods [7-10] is made a brand new direction in foundation of chemistry, substituting classical stoichiometry by linear algebra, from one side and cleaning chemistry from barren intuitionism and its substitution by an elegant formalism from other side. In other words, by this approach is substituted the old chemical particularism by new one mathematical generalism.

Obviously, the continuum problem [146] over chemical equations nocks on chemistry door. This is a very subtle problem, which needs a deeper scientific analysis for its resolution. That kind of problem looks for a logical foundation of chemistry, something similar as foundations of mathematics [147], but it will be a topic of the next research.

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